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Ultra Generalized Metric Space

Orhan GÖÇÜR[1]

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Abstract
In this paper, for n is a natural number such that n→∞, we described ultra generalized metric space as an ultra generalization of a metric space in such a way that the triangle inequality is replaced by similar ones which involve n unit points instead of three. And we give some basic properties about it. For example any ultra generalized metric space may not be a T_1 space.

1. Introduction

In nonlinear analysis, Banach construction principle is a very useful and classical tool (Banach, 1922: 150). Then this principle has been generalized in many directions. For example, an interesting generalization of a metric space introduced by Branciari, by replacing the triangle inequality of a metric space with a similar ones which involve four or more points instead of three (Branciairi, 2000: 32). And then researchers studied fixed point theory in generalized metric space in (Aydi et al, 2012: 46; Mihet, 2009: 92; Samet, 2009: 1265; Samet, 2010: 493).

In this paper, for n is a natural number such that n→∞, we described ultra generalized space as an ultra generalization of a metric space in such a way that the triangle inequality is replaced by similar ones which involve n unit points instead of three. And we give some basic properties about it.

Definition 1 : Let X be a non-empty set and $d: X \times X \rightarrow [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has the following:

(p1) $x = y \Leftrightarrow d(x, y) = 0$,

(p2) $d(x, y) = d(y, x)$,

(p3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. (quadratic inequalty)

[1] Corresponding Author. ORCID: 0000-0001-7141-118X. Asst. Prof. Dr., Bilecik Şeyh Edebali University, orhan.gocur@bilecik.edu.tr

Then, (X, d) is called a generalized metric space (or shortly g.m.s.).

Any metric space is generalized metric space, but the converse is not true (Brianciari, 2000: 32).

2. Main Results

Definition 2 : Let X be a non-empty set and $d: X \times X \to [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $a_1, a_2, \ldots, a_n \in X$, for n is a natural number such that $n \to \infty$, each of them different from x and y, one has the following:

(p1) $d(x, y) = 0 \Leftrightarrow x = y$,

(p2) $d(x, y) = d(y, x)$,

(p3) $d(x, y) \leq d(x, a_1) + d(a_1, a_2) + \ldots + d(a_{n-1}, a_n) + d(a_n, y)$. (circle inequality)

Then, (X, d) is called an ultra generalized metric space (or shortly u.g.m.s.).

Any metric space is generalized metric space, and any generalized metric space is ultra generalized metric space but the converse is not true. We verify this by the following example.

Example 1 : Let $X = A \cup B$, where $A = [0,2]$ and $B = (2, \infty)$. Define the ultra generalized metric d on X as follows:

$$d(x, y) = \begin{cases} |x - y|, & if\ x, y \in B\ or\ x \in A, y \in B\ or\ x \in B, y \in A \\ 1, & if\ \ x \neq y, x, y \in A \\ 0, & if\ x = y, x, y \in A \end{cases}$$

It is clear that d doesn't satisfy the triangle inequality on every where on X. Indeed,

$5 = d(5,0) > d(5,2) + d(2,0) = 3 + 1 = 4$.

And it is clear that d does't satisfy the quadratic inequality on every where on X. Indeed,

$$5 = d(5,0) > d(5,3) + d(3,2) + d(2,0) = 2 + 1 + 1 = 4.$$

Note that d doesn't satisfy quadratic inaquality, triangle inequality or any finite number inaquality nevertheless it satisfy similar ones inaquality which involve n unit points, for n is a natural number such that $n \to \infty$, so d is an ultra generalized metric.

Definition 3 : (X, d) be an ultra generalized metric space and r be a non-negative real number. For any $a \in X$, by an ultra generalized open ball with centre a and radius r, we mean the collection of points of X satisfying $d(x, a) < r$.

The u.g. open ball with centre a and radius r is denoted by $B(a, r)$. Thus $B(a, r) = \{x \in X; d(x, a) < r\}$.

Definition 4 : Let (X, d) be an ultra generalized metric space and r be a non-negative real number. For any $a \in X$, by an ultra generalized closed ball with centre a and radius r, we mean the collection of points of X satisfying $d(x, a) \leq r$.

The u.g. closed ball with centre a and radius r is denoted by $B[a, r]$. Thus $B[a, r] = \{x \in X; d(x, a) \leq r\}$.

Example 2 : Consider the ultra generalized metric space (X, d) as in *Example 1* . Let $a \in X, r$ be a non negative real number. Then

$$B(a, r) = \begin{cases} (\mathrm{a} - \mathrm{r}, \mathrm{a} + \mathrm{r}), & \text{if a} \in \mathrm{B} \\ \{\mathrm{a}\}, & \text{if a} \in \mathrm{A}, \mathrm{r} \leq 1 \\ \mathrm{A}, & \text{if a} \in \mathrm{A}, \mathrm{r} > 1 \end{cases}$$

For example; (0,5)is an u.g. open ball with centre $\left\{\frac{5}{2}\right\}$and radius 5/2

{3/2} is an u.g. open ball with centre $\{\frac{3}{2}\}$ and radius 1

{5/2} is not u.g. open ball and (0,3) is not u.g. open ball.

Also,

$$B[a, r] = \begin{cases} [\mathrm{a} - \mathrm{r}, \mathrm{a} + \mathrm{r}], & \text{if a} \in \mathrm{B} \\ \{\mathrm{a}\}, & \text{if a} \in \mathrm{A}, \mathrm{r} < 1 \\ \mathrm{A}, & \text{if a} \in \mathrm{A}, \mathrm{r} \geq 1 \end{cases}$$

For example; [3,5] is an u.g. closed ball with centre 4 and radius 1

{3/2} is a u.g. closed ball with centre 3/2 and radius 1/2

{5/2} is u.g. closed ball but [1,3] is not u.g. closed ball.

Definition 5 : Let (X, d) be an ultra generalized metric space and $a \in X$. A collection $N(a)$ of points containing a is said to be ultra generalized neighbourhood of a, if there exists a positive real number r such that $a \in B(a, r) \subset N(a)$. Thus $N(a)$ will be called a u.g. neighbourhood of a.

Theorem 1 : Let (X, d) be a u.g.m. space and $a \in X$. Let N_1 and N_2 be u.g. neighbourhood of a in (X, d). Then $N_1 \cap N_2$ is a u.g. neighbourhood of a in (X, d).

Proof : It is obvious from *Definition 3* and *Definition 5*.

Theorem 2 : Every u.g. open ball is an u.g. neighbourhood of each of its points.

Proof : It is obvious from *Definition 3* and *Definition 5*.

Definition 6 : Let (X, d) be an ultra generalized metric space and let A be a subset of X. Then the point a is said to be a interior point of A if there exist $\exists r$, which is a positive real number, such that $a \in B(a, r) \subset A$.

Definition 7 : Let (X, d) be an ultra generalized metric space and let A be a subset of X. Then the interior of A is defined to be the set consisting of all interior points of A.

The interior of A is denoted by $A°$. Hence $A° = \{x \in A; x \in B(x, r) \subset A$, for some positive real number $r\}$.

$A°$ is said to be the interior of A.

Example 3 : Consider the ultra generalized metric space (X, d) as in *Example 1*. Let

$$K = [3,7), \qquad L = (1/2,3/2], \qquad M = (1,5).$$

Then, $K^\circ = (3,7)$

$$L^\circ = \{\{x\}; x \in (1/2,3/2], x \in \mathbb{R}\}$$

$$M^\circ = \{\{x\}; x \in (1,2], x \in \mathbb{R}\} \cup (2,5)$$

Definition 8 : Let (X,d) be an ultra generalized metric space and let A be a non-null subset of X in (X,d). A is said to be an ultra generalized open set in X with respect to d if and only if all points of A be interior point of A.

Thus A° is the largest open set contained in A.

Theorem 3 : In any ultra generalized metric space, every u.g. open ball is an open set.

Proof : It is obvious from *Definition 3* and *Definition 7*.

Theorem 4 : Let (X,d) be an ultra generalized metric space and let A, B be subsets of X. Then

1.$\emptyset^\circ = \emptyset$ and $X^\circ = X$

2.$A^\circ \subset A$,

3.$(A^\circ)^\circ = A^\circ$

4.$A \subset B \;\Rightarrow\; A^\circ \subset B^\circ$

5. $(A \cap B)^\circ = A^\circ \cap B^\circ$

6.$A^\circ \cup B^\circ \subset (A \cup B)^\circ$

Proof :

1.Obvious.

2.Obvious.

3.Since A° is u.g. open and $(A^\circ)^\circ$ is the union of all u.g. open subsets in X contained in $A^\circ, A^\circ \subset (A^\circ)^\circ$. But $(A^\circ)^\circ \subset A^\circ$ by (2). Hence $(A^\circ)^\circ = A^\circ$

4.Suppose that $A \subset B$. Since $A^\circ \subset A \subset B$. A° is an u.g. open subset of B, so by definition of $B^\circ, A^\circ \subset B^\circ$

5.From (4), we have $A \cap B \subset A, A \cap B \subset B$, implies $(A \cap B)^\circ \subset A^\circ, (A \cap B)^\circ \subset B^\circ$so that $(A \cap B)^\circ \subset A^\circ \cap B^\circ$ Also, since $A^\circ \subset A$ and $B^\circ \subset B$ implies $A^\circ \cap B^\circ \subset A \cap B$ so that $A^\circ \cap B^\circ$ is an u.g. open subset of $A \cap B$. Hence $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. Thus $(A \cap B)^\circ = A^\circ \cap B^\circ$.

6.Since $A \subset A \cup B$ and $B \subset A \cup B$. So by (4), $A^\circ \subset (A \cup B)^\circ$ and $B^\circ \subset (A \cup B)^\circ$. So that $A^\circ \cup B^\circ \subset (A \cup B)^\circ$, since $A^\circ \cup B^\circ$ is an u.g. open set.

Remark 1 : Let (X,d) be an ultra generalized metric space and $x, y \in X$. If there exist u.g. open sets A and B such that $x \in A$ and $y \notin A$ or $x \notin B$ and $y \in B$ then X is a T_0 space.

Remark 2: Let (X, d) be an ultra generalized metric space and $x, y \in X$. If there exist u.g. open sets A and B such that $x \in A$ and $y \notin A$ and $x \notin B$ and $y \in B$ then X is a T_1 space.

Remark 3: Let (X, d) be an ultra generalized metric space and $x, y \in X$. If there exist u.g. open sets $x \in A$ and $y \in B$ such that $A \cap B = \emptyset$ then X is a T_2 space (or Hausdorff space).

Propositon 1 : Any ultra generalized metric space may not be a T_2 space.

Example 4 : Let $X = A \cup B$, where $A = [0,2)$ and $B = [2, \infty)$. Define the ultra generalized metric d on X as follows:

$$d(x,y) = \begin{cases} x - y|, & if\ x, y \in B\ or\ x \in A, y \in B\ or\ x \in B, y \in A \\ 1, & if\ \ x \neq y, x, y \in A \\ 0, & if\ x = y, x, y \in A \end{cases}$$

and

$$B(a,r) = \begin{cases} (\mathrm{a} - \mathrm{r}, \mathrm{a} + \mathrm{r}), & \text{if a} \in \mathrm{B} \\ \{\mathrm{a}\}, & \text{if a} \in \mathrm{A}, \mathrm{r} \leq 1 \\ \mathrm{A}, & \text{if a} \in \mathrm{A}, \mathrm{r} > 1 \end{cases}$$

For $0 < r < 1, B(2,r) = (2 - r, 2 + r)$.

There exist open set $\{2 - r/2\}$ and $B(2,r)$ such that $2 - r/2 \in \{2 - r/2\}$ and $2 \in B(2,r)$ and $\{2 - r/2\} \cap (2 - r, 2 + r) = \{2 - r/2\} \neq \emptyset$. This satisfy for all $0 < r < 1$, so X is not a T_2 space.

Proposition 2 : Any ultra generalized metric space may not be a T_1 space.

Example 5 : Consider the ultra generalized metric space (X, d) as in *Example 4.*

for $0 < r < 1, B(2,r) = (2 - r, 2 + r)$.

There exist open set $\{2 - r/2\}$ and $B(2,r)$ such that $2 - r/2 \in \{2 - r/2\}$ and $2 - r/2 \in B(2,r)$ and $2 - r/2 \in B(2,r)$ and $2 \notin \{2 - r/2\}$. This satisfy for all $0 < r < 1$, so X is not a T_1 space.

Also note that X is $T_{\ 0}$ space.

3. Results

In this paper, for n is a natural number such that $n \to \infty$, we described ultra generalized metric space as an ultra generalization of a metric space in such a way that the triangle inequality is replaced by similar ones which involve n unit points instead of three. And we give some basic properties about it. For example any ultra generalized metric space may not be a T_1 space. I hope that the ultra generalized metric space will be used more effectively by researchers. For example in fixed point theorem, constructions etc.

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